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A HIGHER ORDER METHOD OF MULTIPLE SCALES

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1. INTRODUCTION

Perturbation methods are often applied in the analysis of weakly non-linear dynamic systems. The method of multiple scales, for instance, is a common choice. In reference [1], Rahman and Burton proposed a version of the method of multiple scales which can be used to determine the periodic, steady-state primary response of a single-degree-of-freedom, lightly damped, weakly non-linear, forced oscillator. This letter presents some extensions to their method which allows the derivation of modulation equations from which steady state, periodic response solutions and their stability can be determined and allows application to multi-degree-of-freedom systems. Finally, some observations are made which lead to substantial algebraic simplification.

2. ILLUSTRATIVE EXAMPLE

The details of the extended method are illustrated using a Duffing-type equation similar to the example presented in reference [1]:

$$\ddot{u} + \hat{\delta}\dot{u} + u + \hat{\beta}u^3 = \hat{p}\cos\Omega t, \tag{1}$$

where $\hat{\delta}$, $\hat{\beta}$, and $\hat{p}$ are of order ϵ which is a small parameter. The perturbation analysis is detailed here so that the extensions can be clearly pointed out. As in reference [1], steady state periodic solutions are determined for equation (1) to second non-linear order. To begin, new time scales are defined as $T_n = \epsilon^n \Omega t$, $(n = 0, 1, 2, \ldots)$. for a second non-linear order analysis, $n = 0,1,2$. Consequently, equation (1) is recast on the T_0 $(=\Omega t)$ time scale as

$$\Omega^2\ddot{u} + \epsilon\,\delta\Omega\dot{u} + u + \epsilon\beta u^3 = \epsilon p\cos T_0, \tag{2}$$

where the damping, forcing, and cubic non-linear term coefficients have been expanded as $\hat{\delta} = \epsilon\,\delta$, $\hat{p} = \epsilon p$ $(=\epsilon p_1 + \epsilon^2 p_2)$, and $\hat{\beta} = \epsilon\beta$, respectively.

To second non-linear order, the displacement, u, is expanded as

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 \tag{3}$$

and detuning of the excitation frequency is introduced by

$$\Omega^2 = 1 + \epsilon\sigma_1 = 1 + \epsilon\sigma_{11} + \epsilon^2\sigma_{12} \quad \text{and} \quad \Omega = 1 + \epsilon\sigma_{21}. \tag{4}$$

Note that Ω^2 and Ω are expanded *independently*. In this case, Ω is expanded to ϵ order only since it appears in equation (2) as the product $\epsilon\Omega$. The expansion terms σ_{ij} are only mathematical terms used in ordering the expansion and have no physical value. As it will be shown, the solution will always be presented in terms of the original system parameters,

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not the expansion coefficients. This is expected since the final solution should depend only upon the original system parameters.

Substituting the displacement and frequency expansions into equation (2) and collecting like terms of ϵ leads to zeroth, first, and second non-linear order equations:

$$D_0^2 u_0 + u_0 = 0, \tag{5}$$

$$D_0^2 u_1 + u_1 = -2D_0 D_1 u_0 - \sigma_{11} D_0^2 u_0 - \delta D_0 u_0 - \underbrace{\beta u_0^3} + p_1 \cos T_0, \tag{6}$$

$$\begin{aligned} D_0^2 u_2 + u_2 = {} & -2D_0 D_1 u_1 - \underline{D_1^2 u_0} - 2D_0 D_2 u_0 - \sigma_{11}(D_0^2 u_1 + \underline{2D_0 D_1 u_0}) \\ & \underline{\underline{-\sigma_{12} D_0^2 u_0}} - \delta(D_0 u_1 + \underline{D_1 u_0}) - \underline{\underline{\delta\sigma_{21} D_0 u_0}} - \underbrace{3\beta u_0^2 u_1} + \underline{\underline{p_2 \cos T_0}}. \end{aligned} \tag{7}$$

The solution to (5) is $u_0(T_1, T_2) = A(T_1, T_2)\,\mathrm{e}^{\mathrm{i}T_0} + \overline{A}(T_1, T_2)\,\mathrm{e}^{-\mathrm{i}T_0}$ where the overbar stands for the complex conjugate. The slowly varying (complex) amplitude, $A(T_1, T_2) = \frac{1}{2} a\,\mathrm{e}^{\mathrm{i}\theta}$, is determined from the higher order expansions.

Removal of secular terms at the first non-linear order and then at the second non-linear order yields the following equations, respectively:

$$-2\mathrm{i}D_1 A + \sigma_{11} A - \mathrm{i}\delta A - \underbrace{3\beta A^2 \overline{A}} + \tfrac{1}{2} p_1 = 0, \tag{8}$$

$$-\underline{D_1^2 A} - 2\mathrm{i}D_2 A - \underline{2\mathrm{i}\sigma_{11} D_1 A} + \underline{\underline{\sigma_{12} A}} - \underline{\delta D_1 A} - \mathrm{i}\,\underline{\underline{\delta\sigma_{21} A}} - \underbrace{\tfrac{3}{8}\beta^2 A^3 \overline{A}^2} + \tfrac{1}{2}\underline{\underline{p_2}} = 0. \tag{9}$$

These equations can be combined to describe the modulation of the complex amplitude to second non-linear order with respect to the original time scale, t, using

$$\mathrm{d}A/\mathrm{d}t = \epsilon D_1 A + \epsilon^2 D_2 A. \tag{10}$$

In equation (9), the D_1 terms vanish under the assumption that they are independent of the T_2 time scale. Combining equations (8) and (9) in equation (10) yields

$$\begin{aligned} 2\mathrm{i}\frac{\mathrm{d}A}{\mathrm{d}t} &= (\epsilon\sigma_{11} + \epsilon^2\sigma_{12})A - \mathrm{i}(\epsilon\,\delta)(1 + \epsilon\sigma_{21})A + \tfrac{1}{2}(\epsilon p_1 + \epsilon^2 p_2) - 3(\epsilon\beta)A^2\overline{A} - (\epsilon^2\beta^2)\tfrac{3}{8}A^3\overline{A}^2 \\ &= (\Omega^2 - 1)A - \mathrm{i}\,\hat{\delta}\Omega A + \tfrac{1}{2}\hat{p} - 3\hat{\beta}A^2\overline{A} - \tfrac{3}{8}\hat{\beta}^2 A^3 \overline{A}^2, \end{aligned} \tag{11}$$

where all the expansion terms recombine into the original expressions. Note that the only difference between a first non-linear order and a second non-linear order expansion of equation (1) is the term $\frac{3}{8}\hat{\beta}^2 A^3 \overline{A}^2$. The second order expansion has added an $O(\epsilon^2)$ correction arising from the cubic non-linear term. There are no second order correction from the other terms.

Separating the complex amplitude, A, into real and imaginary parts yields the state equations (in polar form):

$$\dot{a} = -\tfrac{1}{2}(\Omega\,\hat{\delta})a - \tfrac{1}{2}\hat{p}\sin\theta, \tag{12}$$

$$a\dot{\theta} = -\tfrac{1}{2}(\Omega^2 - 1)a - \tfrac{1}{2}\hat{p}\cos\theta + \underbrace{\tfrac{3}{8}\hat{\beta}a^3} + \underbrace{\tfrac{3}{256}\hat{\beta}^2 a^5}. \tag{13}$$

Steady-state response can be determined by setting the time derivatives to zero. Combining the above equations leads to

$$(\Omega\,\hat{\delta})^2 a^2 + [-(\Omega^2 - 1)a + \tfrac{3}{4}\hat{\beta}a^3 + \tfrac{3}{128}\hat{\beta}^2 a^5]^2 = \hat{p}^2. \tag{14}$$

This is similar to the equation governing the steady-state response derived in reference [1]. The difference arises from the $\hat{\beta}$ coefficient (which was given as ϵ in reference [1]). The solutions of equation (14) match the numerical integration of the original equation of motion very closely. Note that equation (14) is a fifth order polynomial in a^2 which can only have five, three, or one real root(s). Application of the Routh test shows that equation (14) can only have one or three roots with positive real parts that correspond to a^2;

therefore, for this system, the perturbation analysis does not lead to any spurious responses (more than three roots). Furthermore, higher-order expansions can never introduce spurious responses, since the coefficients of higher-order expansions of the cubic non-linear term will always have the same sign. Thus, there can never be more than three roots with positive real parts. This is in distinct contrast to the spurious responses generated by the multiple scales versions presented in references [2] and [3]. The stability of the solutions can be determined by linearizing equation (11) about each solution and examining the corresponding eigenvalue problem.

3. DISCUSSION

Three key points regarding this procedure are emphasized here: (1) the time scale and all expanded variables are returned to their original form, i.e., the solution is independent of ϵ; (2) time derivative terms are non-zero only on their corresponding time scale e.g., D_1 terms are non-zero on the T_1 scale but vanish on the T_2 scale; and (3) the expansions for Ω^2 and Ω are independent.

It is straightforward to extend this method to multi-degree-of-freedom systems by forming the expression $\mathrm{d}A/\mathrm{d}t$ of equation (10) for each degree of freedom; see references [4, 5] for examples. In both of those cases, the perturbation results closely match the numerical simulations.

Some algebraic simplifications can be employed which greatly reduce the computational effort. At the beginning, all terms including Ω^2, Ω, and $\hat{p}$, are expanded as a power series using ordering terms such as σ_{ij}. However, those terms all recombine into their original form when the expression for $\mathrm{d}A/\mathrm{d}t$ is formulated; see equation (11). They do not have independent contributions on the non-linear time scales. Only the terms related to the cubic non-linearity (identified in equations (6)–(9) and (13) by the underbrace) have distinct non-linear time scale dependent contributions. Therefore, terms involving Ω^2, Ω, and $\hat{p}$ only have to be expanded to first non-linear order. The higher order expansion terms (double underline in equations (7) and (9)) can be neglected since they have no effect on the final recombined equation (11). Furthermore, some derivative terms can also be eliminated. Consider equation (9), the three D_1 terms (underlined) are removed when forming equation (10). Therefore, they can be removed from equation (7) with the *a priori* knowledge that they do not contribute to the steady-state response or to the determination of response stability.

To summarize, the multiple scales method presented in reference [1] can be extended in three ways to (1) develop modulation equations on the original time scale using equation (10) (these equations can be used to determine the stability of the steady-state responses); (2) apply to multi-degree-of-freedom systems; and (3) identify system parameter and derivative terms which do not need to be expanded (to higher non-linear orders) thus saving calculation effort.

It is anticipated that this procedure can be applied successfully to a wide range of multi-degree-of-freedom systems including parametrically excited systems.

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